Sample Problems

1. The sum of two numbers is 31, their difference is 41. Find these numbers.
2. The product of two numbers is 640. Their difference is 12. Find these numbers.
3. One side of a rectangle is 3 ft shorter than twice the other side. Find the sides if the perimeter is 24 ft.
4. One side of a rectangle is 3 ft shorter than twice the other side. Find the sides if the area is 209 ft^2.
5. One side of a rectangle is 4 in shorter than three times the other side. Find the sides if the perimeter of the rectangle is 48 in.
6. One side of a rectangle is 4 in shorter than three times the other side. Find the sides if the area of the rectangle is 319 in^2.
7. We throw an object upward from the top of a 1200 ft tall building. The height of the object, (measured in feet) t seconds after we threw it is
$$h(t) = -16t^2 + 160t + 1200$$
   a) Where is the object 3 seconds after we threw it?
   b) How long does it take for the object to hit the ground?

Practice Problems

1. The product of two numbers is 65. Their difference is 8. Find these numbers.
2. If we square a number, we get six times the number. Find all numbers with this property.
3. If we raise a number to the third power, we get four times the number. Find all numbers with this property.
4. The product of two consecutive even integers is 840. Find these numbers.
5. The area of a rectangle is 1260 m^2. Find the dimensions of the rectangle if we know that one side is 48 m longer than three times the other side.
6. We are standing on the top of a 1680 ft tall building and throw a small object upwards. At every second, we measure the distance of the object from the ground. Exactly t seconds after we threw the object, its height, (measured in feet) is
$$h(t) = -16t^2 + 256t + 1680$$
   a) Find $h(3)$. ($h(3)$ represents the object's position 3 seconds after we threw it.)
   b) How much does the object travel during the two seconds between 5 seconds and 7 seconds?
   c) How long does it take for the object to reach a height of 2640 ft?
   d) How long does it take for the object to hit the ground?

Sample Problems - Answers

1. -5 and 36
2. 20 with 32 and -32 with -20
3. $5\,\text{ft}$ by $7\,\text{ft}$
4. $11\,\text{ft}$ and $19\,\text{ft}$
5. $7\,\text{in}$ by $17\,\text{in}$
6. $11\,\text{in}$ by $29\,\text{in}$
7. a) $1536\,\text{ft}$ b) 15 seconds

Practice Problems - Answers

1. 5 with 13 and -13 with -5
2. $0, 6$
3. $-2, 0, 2$
4. $28, 30$ and $-30, -28$
5. $14\,\text{m}$ by $90\,\text{m}$
6. a) $2304\,\text{ft}$ b) $128\,\text{ft}$ c) 6 seconds and 10 seconds d) 21 seconds

Sample Problems - Solutions

1. The sum of two numbers is 31, their difference is 41. Find these numbers.
Solution: Let us denote the smaller number by x. Then the larger number is $x+41$, since the difference between the two numbers is 41. The equation then is

$$\begin{aligned}
\underbrace{x}_{\text{smaller number}} + \underbrace{x+41}_{\text{larger number}} &= 31 && \text{solve for } x \\
2x+41 &= 31 && \text{subtract } 41 \\
2x &= -10 && \text{divide by } 2 \\
x &= -5
\end{aligned}$$

Thus the smaller number, labeled x is -5. The larger number was labeled $x+41$, so it must be $-5+41=36$. Thus the numbers are -5 and 36. We check: the difference between 36 and -5 is $36-(-5)=41$, and their sum is indeed $36+(-5)=31$. Thus our solution is correct.

2. The product of two numbers is 640. Their difference is 12. Find these numbers.
Solution: Let us label the smaller number as x. Then the larger number is $x+12$. The equation is

$$\begin{aligned}
x(x+12) &= 640 && \text{Solve for } x \\
x^2+12x &= 640 \\
x^2+12x-640 &= 0
\end{aligned}$$

We will solve this quadratic equation by completing the square. Half of the linear coefficient is 6.

$$\begin{aligned}
x^2+12x-640 &= 0 && (x+6)^2 = x^2+12x+36 \\
\underbrace{x^2+12x+36}-36-640 &= 0 \\
(x+6)^2-676 &= 0 && \sqrt{676}=26 \\
(x+6)^2-26^2 &= 0 \\
(x+6+26)(x+6-26) &= 0 \\
(x+32)(x-20) &= 0 && \Longrightarrow \quad x_1=-32 \text{ and } x_2=20
\end{aligned}$$

If $x=-32$, then the larger number is $-32+12=-20$. If $x=20$, then the larger number is $20+12=32$. **The two solutions of the equation do not determine a pair of numbers: they are the smaller numbers in two pairs!** The answer is: 20 with 32 and -32 with -20. We check in both cases: with 20 and 32

$$20(32)=640 \quad \text{and} \quad 32-20=12$$

and with -32 and -20

$$-32(-20)=640 \quad \text{and} \quad -20-(-32)=12$$

3. One side of a rectangle is 3 ft shorter than twice the other side. Find the sides if the perimeter is 24 ft.
Solution: Let us denote the shorter side by x. Then the longer side is $2x-3$. The equation expresses the perimeter.

$$\begin{aligned}
24 &= 2(x)+2(2x-3) && \text{Solve for } x \\
24 &= 2x+4x-6 && \text{combine like terms} \\
24 &= 6x-6 && \text{add } 6 \\
30 &= 6x && \text{divide by } 6 \\
5 &= x
\end{aligned}$$

Thus the shorter side is 5 ft, and the longer side is $2(5)-3=7$ ft. Thus the answer is: 5 ft by 7 ft. We check: 7 is indeed 3 less than twice 5, i.e. $7=2(5)-3$ and the perimeter is $2(5\,\text{ft})+2(7\,\text{ft})=24\,\text{ft}$. Thus our solution is indeed correct.

4. One side of a rectangle is 3 ft shorter than twice the other side. Find the sides if the area is $209\,\text{ft}^2$.
Solution: Let us denote the shorter side by x. Then the longer side is $2x-3$. The equation expresses the area.

$$\begin{aligned} x(2x-3) &= 209 \qquad \text{solve for } x \\ 2x^2-3x &= 209 \\ 2x^2-3x-209 &= 0 \end{aligned}$$

Since this is a quadratic equation, our only method of solving it is by the zero product rule. There are several factoring techniques available, we will present two of them.
Method 1. We will factor the expression $2x^2-3x-209$ by completing the square. First we have to factor out the leading coefficient. (For more on this, see the handout on completing the square part 3.)

$$\begin{aligned} 2x^2-3x-209 &= 0 \qquad \text{factor out } 2 \\ 2\left(x^2-\frac{3}{2}x-\frac{209}{2}\right) &= 0 \end{aligned}$$

We will factor the expression in the parentheses now. Half of the linear coefficient is $\frac{3}{2}\div 2=\frac{3}{2}\cdot\frac{1}{2}=\frac{3}{4}$ and so our complete square is

$$\left(x-\frac{3}{4}\right)^2=x^2-\frac{3}{4}x-\frac{3}{4}x+\frac{9}{16}=x^2-\frac{3}{2}x+\frac{9}{16}$$

Thus we smuggle in $\frac{9}{16}$.

$$\begin{aligned} 2\left(x^2-\frac{3}{2}x-\frac{209}{2}\right) &= 0 \\ 2\left(\underbrace{x^2-\frac{3}{2}x+\frac{9}{16}}-\frac{9}{16}-\frac{209\cdot 8}{2\cdot 8}\right) &= 0 \\ 2\left(\left(x-\frac{3}{4}\right)^2-\frac{9}{16}-\frac{1672}{16}\right) &= 0 \\ 2\left(\left(x-\frac{3}{4}\right)^2-\frac{1681}{16}\right) &= 0 \qquad \sqrt{\frac{1681}{16}}=\frac{\sqrt{1681}}{\sqrt{16}}=\frac{41}{4} \\ 2\left(\left(x-\frac{3}{4}\right)^2-\left(\frac{41}{4}\right)^2\right) &= 0 \\ 2\left(x-\frac{3}{4}+\frac{41}{4}\right)\left(x-\frac{3}{4}-\frac{41}{4}\right) &= 0 \\ 2\left(x+\frac{38}{4}\right)\left(x-\frac{44}{4}\right) &= 0 \\ 2\left(x+\frac{19}{2}\right)(x-11) &= 0 \quad \Longrightarrow \quad x_1=-\frac{19}{2} \text{ and } x_2=11 \end{aligned}$$

Method 2. We will factor the expression $2x^2-3x-209$ by grouping. First we need to find two numbers, p and q such that

$$\begin{aligned} pq &= -418 \qquad \text{(product of 1st and 3rd coefficient)} \\ p+q &= -3 \qquad \text{(second coefficient)} \end{aligned}$$

If the product pq has to be large but the sum is relatively small, like in this case, we should start looking close to $\sqrt{418}$, which is approximately $\sqrt{418} = 20.445$. We find that 19 and -22 work. We factor by grouping.

$$\begin{aligned} 2x^2 - 3x - 209 &= 0 \\ \underbrace{2x^2 + 19x}\underbrace{-22x - 209} &= 0 \\ x(2x+19) - 11(2x+19) &= 0 \\ (x-11)(2x+19) &= 0 \quad \Longrightarrow \quad x_1 = 11 \quad \text{and} \quad x_2 = -\frac{19}{2} \end{aligned}$$

Since x denotes the side of a rectangle, which is a distance, and **distances are never negative**, the second solution, $x_2 = -\dfrac{19}{2}$ is immediately ruled out. If $x = 11$, the other side must be $2(11) - 3 = 19$. Thus the sides of the rectangle are 11 ft and 19 ft. We check:

$$\begin{aligned} 2(11) - 3 &= 19 \checkmark \quad \text{and} \\ 11(19) &= 209 \checkmark \end{aligned}$$

Thus our solution is indeed correct.

5. One side of a rectangle is 4 in shorter than three times the other side. Find the sides if the perimeter of the rectangle is 48 in.
Solution: Let us denote the shorter side by x. Then the other side is $3x - 4$. The equation expresses the perimeter of the rectangle.

$$\begin{aligned} 2(x) + 2(3x-4) &= 48 && \text{multiply out parentheses} \\ 2x + 6x - 8 &= 48 && \text{combine like terms} \\ 8x - 8 &= 48 && \text{add} \\ 8x &= 56 && \text{divide by 8} \\ x &= 7 \end{aligned}$$

If the shorter side was denoted by x, we now know it is 7 in. The longer side was denoted by $3x - 4$, so it must be $3(7) - 4 = 17$ inches. Thus the sides of the rectangle are 7 in and 17 in. We check: $P = 2(7\,\text{in}) + 2(17\,\text{in}) = 48\,\text{in}$ and $17\,\text{in} = 3(7\,\text{in}) - 4\,\text{in}$. Thus our solution is correct.

6. One side of a rectangle is 4 in shorter than three times the other side. Find the sides if the area of the rectangle is $319\,\text{in}^2$.
Solution: Let us denote the shorter side by x. Then the other side is $3x - 4$. The equation expresses the area of the rectangle.

$$\begin{aligned} x(3x-4) &= 319 && \text{multiply out parentheses} \\ 3x^2 - 4x &= 319 && \text{subtract 319} \\ 3x^2 - 4x - 319 &= 0 \end{aligned}$$

Because the equation is quadratic, we need to factor the left-hand side and then apply the zero property. We will factor by grouping, also known as the AC-method. We are looking for numbers p and q with the following conditions. The sum of p and q has to be the linear coefficient (the number in front of x, with its sign), so it is -4. The product of p and q has to be the product of the other coefficients, $3(-319) = -957$.

$$\begin{aligned} pq &= -957 \\ p + q &= -4 \end{aligned}$$

Now we need to find p and q. Because the product is negative, we re looking for a positive and a negative number. Because the sum is negtive, the larger number must carry the negative sign. We enter $\sqrt{957}$ into the calculator and get a decimal:

$$\sqrt{957} = 30.935....$$

So we start looking for factors of 957, starting at 30, and moving down. We soon find 29 and -33. These are our values for p and q. We use these numbers to express the linear term:

$$-4x = 29x - 33x$$

and factor by grouping.

$$\begin{aligned} 3x^2 - 4x - 319 &= 0 \\ \underbrace{3x^2 + 29x}\,\underbrace{-33x - 319} &= 0 \\ x(3x+29) - 11(3x+29) &= 0 \\ (x-11)(3x+29) &= 0 \qquad \Longrightarrow \qquad x_1 = 11 \text{ and } x_2 = -\frac{29}{3} \end{aligned}$$

Since distances can not be negative, the second solution for x, $-\dfrac{29}{3}$ is ruled out. Thus $x = 11$. Then the longer side is $3(11) - 4 = 29$, and so the rectangle's sides are $11\,\text{in}$ and $29\,\text{in}$ long. We check: $11\,\text{in}\,(29\,\text{in}) = 319\,\text{in}^2$ and $29\,\text{in} = 3(11\,\text{in}) - 4\,\text{in}$. Thus our solution is correct.

7. We throw an object upward from the top of a 1200 ft tall building. The height of the object, (measured in feet) t seconds after we threw it is

$$h(t) = -16t^2 + 160t + 1200$$

a) Where is the object 3 seconds after we threw it?
Solution: We need to compute $h(3)$. This means that we substitute 3 into t and evaluate the algebraic expression.

$$\begin{aligned} h(3) &= -16 \cdot 3^2 + 160 \cdot 3 + 1200 = -16 \cdot 9 + 160 \cdot 3 + 1200 \\ &= -144 + 480 + 1200 = 336 + 1200 = 1536 \end{aligned}$$

Thus the object is 1536 ft high after 3 seconds.

b) How long does it take for the object to hit the ground?
Solution: we need to solve the equation $t = ?$ so that $h(t) = 0$

$$\begin{aligned} h(t) &= 0 \\ -16t^2 + 160t + 1200 &= 0 \qquad \text{factor out } -16 \\ -16\left(t^2 - 10t - 75\right) &= 0 \end{aligned}$$

We will factor $t^2 - 10t + 75$ by completing the square.

$$\begin{aligned} -16\left(t^2 - 10t - 75\right) &= 0 && (t-5)^2 = t^2 - 10t + 25 \quad \text{smuggle in } 25 \\ -16\left(\underbrace{t^2 - 10t + 25} - 25 - 75\right) &= 0 \\ -16\left((t-5)^2 - 100\right) &= 0 && \text{re-write 100 as } 10^2 \\ -16\left((t-5)^2 - 10^2\right) &= 0 && \text{factor via the difference of squares theorem} \\ -16(t-5+10)(t-5-10) &= 0 && \text{simplify} \\ -16(t+5)(t-15) &= 0 && \text{apply zero property} \end{aligned}$$

$$t = -5 \quad \text{or} \quad t = 15$$

Since the negative solution, $t = -5$ does not make sense in the context of the problem, it is ruled out. We check $t = 15$:

$$\begin{aligned} h(3) &= -16 \cdot 15^2 + 160 \cdot 15 + 1200 \\ &= -16 \cdot 225 + 160 \cdot 15 + 1200 \\ &= -3600 + 2400 + 1200 \\ &= -1200 + 1200 = 0 \end{aligned}$$

Thus the answer is: 15 seconds.

For more documents like this, visit our page at http://www.teaching.martahidegkuti.com and click on Lecture Notes. E-mail questions or comments to mhidegkuti@ccc.edu.